Oral Exam for Individuals: Applied and Computational Mathematics 2017

1. Consider the following Burgers' equation:

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & x \in R, \quad t > 0 \\ u(x,0) = u_0(x), & x \in R\,, \end{cases} \tag{1}$$

which can be used to model the motion of a viscous compressible gas, where $u(x,t)$ is the speed of the gas, $\nu > 0$ is the kinematic viscosity, x is the spatial coordinate, and t is the time. As it is shown below, we can apply the Hopf-Cole transformation to solve the strongly nonlinear Burgers equation (1).

   (a) Let $U_x = u$ and introduce the Hopf-Cole transformation $U(x,t) = -2\nu \log(\phi(x,t))$. Derive the equation for $\phi(x,t)$.

   (b) Solve the equation for $\phi(x,t)$, and then obtain the solution to the Burgers equation (1).

   (c) When $\nu = 0$ the Burgers' equation (1) becomes the inviscid Burgers' equation. Show that if we solve the inviscid Burgers' equation with smooth initial data $u_0(x)$, for which $u_0'(x)$ is negative and bounded from below, then the solution will break at time $T_b = \frac{-1}{\min u_0'(x)}$. The term "break" here means the solution $u(x,t)$ has an infinite slope and a shock form.

2. (a) Consider the following Rudin-Osher-Fatemi model for image regularization

$$\min_u \int_\Omega \|\nabla u\| + \lambda\,(F - u)^2 \tag{2}$$

   for some given function F in a boundary rectangular domain Ω, where $\|\cdot\|$ is the usual 2-norm. Show that the Euler-Lagrange equation corresponding to the variational problem (2) is given by

$$u - \frac{1}{2\lambda}\nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|}\right) = F\,.$$

   (b) Consider the following variational problem for the cartoon-texture decomposition in image processing

$$\min_{u,g,h} \int_\Omega \|\nabla u\| + \lambda\left(f - \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} - u\right)^2 + \mu\sqrt{g^2 + h^2}\,, \tag{3}$$

   where f is the given image, λ and μ are two positive constants, and $f, u, g, h : \Omega \to \mathbb{R}$. With appropriate boundary conditions, derive the system of Euler-Lagrangian equations for (3) that the minimizers u, g and h should satisfy.

   (c) Let $\kappa(u) = \nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|}\right)$ be the curvature operator. If u solves the variational problem (3), show that $\|\nabla \kappa(u)\| = \mu$.